HOMOGENEOUS EINSTEIN METRICS ON THE GENERALIZED FLAG MANIFOLD $Sp(n)/(U(p)\times U(n-p))$

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Abstract. We find the precise number of non-Kähler $Sp(n)$-invariant Einstein metrics on the generalized flag manifold $M = Sp(n)/(U(p)\times U(n-p))$ with $n\geq 3$ and $1\leq p\leq n-1$. We use an analysis on parametric systems of polynomial equations and we give some insight towards the study of such systems.

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Introduction

A Riemannian metric g is called *Einstein* if the Ricci tensor Ric_g satisfies the equation $\mathrm{Ric}_g = e\cdot g$, for some $e\in\mathbb{R}$. When M is compact, Einstein metrics of volume 1 can be characterized variationally as the critical points of the scalar curvature functional $T(g)=\int_M S_g d\mathrm{vol}_g$ on the space $\mathcal{M}_1$ of Riemannian metrics of volume 1. If $M=G/K$ is a compact homogeneous space, a G-invariant Einstein metric is precisely a critical point of T restricted to the set of G-invariant metrics of volume 1. As a consequence, the Einstein equation reduces to a system of non-linear algebraic equations, which is still very complicated but more manageable, and sometimes can be solved explicity. Thus most known examples of Einstein manifolds are homogeneous.

A generalized flag manifold is an adjoint orbit of a compact semisimple Lie group G, or equivalently a compact homogeneous space of the form $M=G/K=G/C(S)$, where $C(S)$ is the centralizer of a torus S in G. Einstein metrics on generalized flag manifolds have been studied by several authors (Alekseevsky, Arvanitoyeorgos, Kimura, Sakane, Chrysikos, Negreiros).

Eventhough the problem of finding all invariant Einstein metrics on M can be facilitated by use of certain theoretical results (e.g. the work [Grv] on the total number of G-invariant complex Einstein metrics), it still remains a difficult one, especially when the number of isotropy summands increases. This difficulty also increases when we pass from exceptional flag manifolds to classical flag manifolds, because in the later case the Einstein equation reduces to a parametric system.

In two recent works [AC] and [ACS] all invariant Einstein metrics were found for all generalized flag manifolds with four isotropy summands, but a partial answer was given for the space $Sp(n)/(U(p)\times U(n-p))$.

We summarize the results obtained about this space.

Theorem 1. ([AC]) *The flag manifold $Sp(n)/(U(p)\times U(n-p))$ ($n\geq 2$ and $1\leq p\leq n-1$) admits at least four $Sp(n)$-invariant Einstein metrics, which are Kähler.*

Theorem 2. ([AC]) *The flag manifold $Sp(2p)/(U(p)\times U(p))$ ($p\geq 1$) admits precisely six $Sp(2p)$-invariant Einstein metrics. There are four isometric Kähler-Einstein metrics, and two non-Kähler Einstein metrics.*

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In the present paper we find all $Sp(n)$-invariant Einstein metrics on the flag manifold $Sp(n)/(U(p)\times U(n-p))$, by using an approach similar to the one used in [ACS].

The Einstein equation reduces to polynomial systems whose coefficients involve parameters, so a demanding task is to show existence and uniqueness of solutions of such systems.

Our main result is the following:

Main Theorem. *The generalized flag manifold $M = Sp(n)/(U(p)\times U(n-p))$ with $n\geq 3$ and $1\leq p\leq n-1$, admits precisely two non-Kähler $Sp(n)$-invariant Einstein metrics.*

1. The Einstein equation for flag manifolds

Let $M = G/K = G/C(S)$ be a generalized flag manifold of a compact simple Lie group G, where $K = C(S)$ is the centralizer of a torus S in G. Let $o = eK$ be the identity coset of G/K. We denote by $\mathfrak{g}$ and $\mathfrak{k}$ the corresponding Lie algebras of G and K. Let B denote the Killing form of $\mathfrak{g}$. Since G is compact and simple, $-B$ is a positive definite inner product on $\mathfrak{g}$. With repsect to $-B$ we consider the orthogonal decomposition $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{m}$. This is a reductive decomposition of $\mathfrak{g}$, that is $\mathrm{Ad}(K)\mathfrak{m}\subset\mathfrak{m}$, and as usual we identify the tangent space T_oM with $\mathfrak{m}$. Since $K = C(S)$, the isotropy group K is connected and the relation $\mathrm{Ad}(K)\mathfrak{m}\subset\mathfrak{m}$ is equivalent with $[\mathfrak{k},\mathfrak{m}]\subset\mathfrak{m}$. Thus, for a flag manifold $M = G/K$ the notion of $\mathrm{Ad}(K)$-invariant and $\mathrm{ad}(\mathfrak{k})$-invariant is equivalent.

Let $\chi : K\to \mathrm{Aut}(T_oM)$ be the isotropy representation of K on T_oM. Since χ is equivalent to the adjoint representation of K restricted on $\mathfrak{m}$, the set of all G-invariant symmetric covariant 2-tensors on G/K can be identified with the set of all $\mathrm{Ad}(K)$-invariant symmetric bilinear forms on $\mathfrak{m}$. In particular, the set of G-invariant metrics on G/K is identified with the set of $\mathrm{Ad}(K)$-invariant inner products on $\mathfrak{m}$.

Let $\mathfrak{m} = \mathfrak{m}_1\oplus\cdots\oplus\mathfrak{m}_s$ be a $(-B)$-orthogonal $\mathrm{Ad}(K)$-invariant decomposition of $\mathfrak{m}$ into pairwise inequivalent irreducible $\mathrm{Ad}(K)$-modules $\mathfrak{m}_i$ $(i = 1,\dots,s)$. Such a decomposition always exists and can be expressed in terms of $\mathfrak{t}$-roots (cf. [AP], [AC]). Then, a G-invariant Riemannian metric on M (or equivalently, an $\mathrm{Ad}(K)$-invariant inner product $\langle\ ,\ \rangle$ on $\mathfrak{m} = T_oM$) is given by

$$g = \langle\ ,\ \rangle = x_1\cdot(-B)|_{\mathfrak{m}_1} + \cdots + x_s\cdot(-B)|_{\mathfrak{m}_s}, \tag{1}$$

where $(x_1,\dots,x_s)\in\mathbb{R}^s_+$. Since $\mathfrak{m}_i\neq\mathfrak{m}_j$ as $\mathrm{Ad}(K)$-representations, any G-invariant metric on M has the above form.

Similarly, the Ricci tensor Ric_g of a G-invariant metric g on M, as a symmetric covariant 2-tensor on G/K is given by

$$\mathrm{Ric}_g = r_1x_1\cdot(-B)|_{\mathfrak{m}_1} + \cdots + r_sx_s\cdot(-B)|_{\mathfrak{m}_s},$$

where $r_1,\dots,r_s$ are the components of the Ricci tensor on each $\mathfrak{m}_i$, that is $\mathrm{Ric}_g|_{\mathfrak{m}_i} = r_ix_i\cdot(-B)|_{\mathfrak{m}_i}$. These components have o useful description in terms of the structure constants $[ijk]$ first introduced in [WZ]. Let $\{X_\alpha\}$ be a $(-B)$-orthonormal basis adapted to the decomposition of $\mathfrak{m}$, that is $X_\alpha\in\mathfrak{m}_i$ for some i, and $\alpha<\beta$ if $i<j$ (with $X_\alpha\in\mathfrak{m}_i$ and $X_\beta\in\mathfrak{m}_j$). Set $A^\gamma_{\alpha\beta} = B([X_\alpha,X_\beta],X_\gamma)$ so that $[X_\alpha,X_\beta]_{\mathfrak{m}} = \sum_\gamma A^\gamma_{\alpha\beta}X_\gamma$, and $[ijk] = \sum(A^\gamma_{\alpha\beta})^2$, where the sum is taken over all indices α,β,γ with $X_\alpha\in\mathfrak{m}_i, X_\beta\in\mathfrak{m}_j, X_\gamma\in\mathfrak{m}_k$ (where $[\ ,\ ]_{\mathfrak{m}}$ denotes the $\mathfrak{m}$-component). Then $[ijk]$ is nonnegative, symmetric in all three entries, and independent of the $(-B)$-orthonormal

bases choosen for $\mathfrak{m}_i, \mathfrak{m}_j$ and $\mathfrak{m}_k$ (but it depends on the choise of the decomposition of $\mathfrak{m}$).

Proposition 1. ([PaS]) *Let $M = G/K$ be a generalized flag manifold of a compact simple Lie group G and let $\mathfrak{m} = \bigoplus_{i=1}^{s} \mathfrak{m}_i$ be a decomposition of $\mathfrak{m}$ into pairwise inequivalent irreducible* $\mathrm{Ad}(K)$*-submodules. Then the components $r_1, \ldots, r_s$ of the Ricci tensor of a G-invariant metric* (1) on M are given by

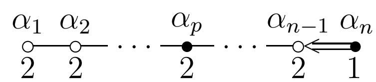
$$r_k = \frac{1}{2x_k} + \frac{1}{4d_k}\sum_{i,j}\frac{x_k}{x_i x_j}[ijk] - \frac{1}{2d_k}\sum_{i,j}\frac{x_j}{x_k x_i}[kij], \qquad (k = 1, \ldots, s).$$

In wiew of Proposition 1, a G-invariant metric $g = (x_1, \ldots, x_s) \in \mathbb{R}^s_+$ on M, is an Einstein metric with Einstein constant e, if and only if it is a positive real solution of the system

$$\frac{1}{2x_k} + \frac{1}{4d_k}\sum_{i,j}\frac{x_k}{x_i x_j}[ijk] - \frac{1}{2d_k}\sum_{i,j}\frac{x_j}{x_k x_i}[kij] = e, \quad 1 \le k \le s.$$

2. The generalized flag manifold $Sp(n)/(U(p)\times U(n-p))$

We review some results related to the generalized flag manifold $M = G/K = Sp(n)/(U(p)\times U(n-p))$ ($n \geq 3,\ 1 \leq p \leq n-1$) obtained in [AC]. Its corresponding painted Dynkin diagram is given by



The isotropy representation of M decomposes into a direct sum $\chi = \chi_1 \oplus \chi_2 \oplus \chi_3 \oplus \chi_4$, which gives rise to a decomposition $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3 \oplus \mathfrak{m}_4$ of $\mathfrak{m} = T_o M$ into four irreducible inequivalent $\mathrm{ad}(\mathfrak{k})$-submodules. The dimensions $d_i = \dim \mathfrak{m}_i$ ($i = 1, 2, 3, 4$) of these submodules can be obtained by use of Weyl's formula [AC, pp. 204-205, p. 210] and are given by

$$d_1 = 2p(n-p),\ d_2 = (n-p)(n-p+1),\ d_3 = 2p(n-p),\ d_4 = p(p+1).$$

According to (1), a G-invariant metric on $M = G/K$ is given by

$$\langle\ ,\ \rangle = x_1 \cdot (-B)|_{\mathfrak{m}_1} + x_2 \cdot (-B)|_{\mathfrak{m}_2} + x_3 \cdot (-B)|_{\mathfrak{m}_3} + x_4 \cdot (-B)|_{\mathfrak{m}_4}, \tag{2}$$

for positive real numbers x_1, x_2, x_3, x_4. We will denote such metrics by $g = (x_1, x_2, x_3, x_4)$.

It is known ([Nis]) that if $n \neq 2p$ then M admits two non-equivalent G-invariant complex structures J_1, J_2, and thus two non-isometric Kähler-Einstein metrics which are given (up to scale) by (see also [AC, Theorem 3])

$$\begin{aligned} g_1 &= (n/2,\ n+p+1,\ n/2+p+1,\ p+1) \\ g_2 &= (n/2,\ n-p+1,\ 3n/2-p+1,\ 2n-p+1). \end{aligned} \tag{3}$$

If $n = 2p$ then M admits a unique G-invariant complex structure with corresponding Kähler-Einstein metric (up to scale) $g = (p,\ p+1,\ 2p+1,\ 3p+1)$ (cf. also [AC, Theorem 10] where all isometric Kähler-Einstein metrics are listed).

The Ricci tensor of M is given as follows:

Proposition 2. *(*[AC]*) The components r_i of the Ricci tensor for a G-invariant Riemannian metric on M determined by* (2) *are given as follows:*

$$\left.\begin{aligned}
r_1 &= \frac{1}{2x_1} + \frac{c_{12}^3}{2d_1}\Big(\frac{x_1}{x_2x_3} - \frac{x_2}{x_1x_3} - \frac{x_3}{x_1x_2}\Big) + \frac{c_{13}^4}{2d_1}\Big(\frac{x_1}{x_3x_4} - \frac{x_4}{x_1x_3} - \frac{x_3}{x_1x_4}\Big)\\
r_2 &= \frac{1}{2x_2} + \frac{c_{12}^3}{2d_2}\Big(\frac{x_2}{x_1x_3} - \frac{x_1}{x_2x_3} - \frac{x_3}{x_1x_2}\Big)\\
r_3 &= \frac{1}{2x_3} + \frac{c_{12}^3}{2d_3}\Big(\frac{x_3}{x_1x_2} - \frac{x_2}{x_1x_3} - \frac{x_1}{x_2x_3}\Big) + \frac{c_{13}^4}{2d_3}\Big(\frac{x_3}{x_1x_4} - \frac{x_4}{x_1x_3} - \frac{x_1}{x_3x_4}\Big)\\
r_4 &= \frac{1}{2x_4} + \frac{c_{13}^4}{2d_4}\Big(\frac{x_4}{x_1x_3} - \frac{x_3}{x_1x_4} - \frac{x_1}{x_3x_4}\Big),
\end{aligned}\right\} \tag{4}$$

where $c_{12}^3 = [123]$ *and* $c_{13}^4 = [134]$.

By taking into account the explicit form of the Kähler-Einstein metrics above, and substituting these in (4), we find that the values of the unknown triples $[ijk]$ are $c_{12}^3 = \dfrac{p(n-p)(n-p+1)}{2(n+1)}$ and $c_{13}^4 = \dfrac{p(p+1)(n-p)}{2(n+1)}$.

A G-invariant metric $g = (x_1, x_2, x_3, x_4)$ on $M = G/K$ is Einstein if and only if, there is a positive constant e such that $r_1 = r_2 = r_3 = r_4 = e$, or equivalently

$$r_1 - r_3 = 0, \quad r_1 - r_2 = 0, \quad r_3 - r_4 = 0. \tag{5}$$

By substituting the values of d_i $(i = 1, 2, 3, 4)$ and c_{12}^3, c_{13}^4 into the components of the Ricci tensor, then System (5) is equivalent to the following equations:

$$\left.\begin{aligned}
(x_1 - x_3)(x_1x_2 + px_1x_2 + x_2x_3 + px_2x_3 + x_1x_4 + nx_1x_4\\
-px_1x_4 - 2x_2x_4 - 2nx_2x_4 + x_3x_4 + nx_3x_4 - px_3x_4) = 0\\
4(n+1)x_3x_4(x_2 - x_1) + (n+p+1)x_4(x_1^2 - x_2^2) - (n-3p+1)x_3^2x_4\\
+(p+1)x_2(x_1^2 - x_3^2 - x_4^2) = 0\\
4(n+1)x_1x_2(x_4 - x_3) + (2n-p+1)x_2(x_3^2 - x_4^2) + (2n-3p-1)x_1^2x_2\\
+(n-p+1)x_4(x_3^2 - x_1^2 - x_2^2) = 0
\end{aligned}\right\} \tag{6}$$

3. Proof of the Main Theorem

Consider the equation $r_1 - r_3 = 0$. Then we have

$$\begin{gathered}
(x_1 - x_3)(x_1x_2 + px_1x_2 + x_2x_3 + px_2x_3 + x_1x_4 + nx_1x_4\\
-px_1x_4 - 2x_2x_4 - 2nx_2x_4 + x_3x_4 + nx_3x_4 - px_3x_4) = 0.
\end{gathered}$$

We claim that, for $x_1 = x_3$, there are no Einstein metrics, and that, for the other case, there exist Einstein metrics on $Sp(n)/(U(p) \times U(n-p))$.

Case 1. For $x_1 = x_3$, we put $x_1 = 1$ and we get the following system of equations

$$(n+p+1){x_2}^2 + 4(n-p+1) - 4(n+1)x_2 + (p+1)x_2x_4 = 0, \tag{7}$$
$$(n-p+1)x_2x_4 + (2n-p+1){x_4}^2 - 4(n+1)x_4 + 4(p+1) = 0. \tag{8}$$

From equation (8), we have

$$x_2 = \frac{-(2n-p+1){x_4}^2 + 4(n+1)x_4 - 4(p+1)}{x_4(n-p+1)}. \tag{9}$$

Now we substitute equation (9) into the equation (7), and we obtain the following equation:

$$
\begin{aligned}
&f_{n,p}(x_4) = n(n+1)(2n-p+1){x_4}^4 - 4(n+1)\left(n^2+2np+n-p^2+p\right){x_4}^3 \\
&+2\left(n^3+9n^2p+7n^2+4np^2+16np+8n-2p^3+2p^2+6p+2\right){x_4}^2 \\
&-8(n+1)(p+1)(n+3p+1)x_4 + 8(p+1)^2(n+p+1) = 0.
\end{aligned}
$$

From equation (8), we have

$$
x_4 = \frac{-(n+p+1){x_2}^2 + 4(n+1)x_2 - 4(n-p+1)}{(p+1)x_2}. \tag{10}
$$

Now we substitute equation (10) into equation (7), and we obtain the following equation:

$$
\begin{aligned}
&g_{n,p}(x_2) = n(n+1)(n+p+1){x_2}^4 - 4(n+1)\left(2n^2+2n-p^2-p\right){x_2}^3 \\
&+2\left(12n^3-11n^2p+25n^2-2np^2-20np+14n+2p^3+2p^2-6p+2\right){x_2}^2 \\
&-8(n+1)x_2(4n-3p+1)(n-p+1) + 8(n-p+1)^2(2n-p+1) = 0.
\end{aligned}
$$

We claim that, for $n/2 \leq p \leq n-1$, there are no positive solutions of the equation $f_{n,p}(x_4) = 0$. Note that $g_{n,p}(x_4) = f_{n,n-p}(x_4)$, thus we also see that, for $1 \leq p \leq n/2$, there are no positive solutions of the equation $g_{n,p}(x_2) = 0$.

It is

$$
\begin{aligned}
&\frac{df_{n,p}}{dx_4}(x_4) = 4n(n+1)(2n-p+1){x_4}^3 - 12(n+1)\left(n^2+2np+n-p^2+p\right){x_4}^2 \\
&+4\left(n^3+9n^2p+7n^2+4np^2+16np+8n-2p^3+2p^2+6p+2\right)x_4 \\
&-8(n+1)(p+1)(n+3p+1).
\end{aligned}
$$

Note that the coefficient of ${x_4}^3$ is $4n(n+1)(2n-p+1) > 0$.

By evaluating $\dfrac{df_{n,p}}{dx_4}(x_4)$ at $x_4 = \dfrac{2(p-1)}{n}$, we have

$$
\frac{df_{n,p}}{dx_4}\left(\frac{2(p-1)}{n}\right) = -\frac{8(n-p+1)\left(2n^3-3n^2p+13n^2-8np+16n+2p^3-6p+4\right)}{n^2}.
$$

Since we can write

$$
\begin{aligned}
&2n^3-3n^2p+13n^2-8np+16n+2p^3-6p+4 \\
&= 2(n-p)^3 + (3p+13)(n-p)^2 + (18p+16)(n-p) + p^3 + 5p^2 + 10p + 4,
\end{aligned}
$$

we see that $\dfrac{df_{n,p}}{dx_4}\left(\dfrac{2(p-1)}{n}\right) < 0$.

By evaluating $\dfrac{df_{n,p}}{dx_4}(x_4)$ at $x_4 = \dfrac{2(p+1)}{n}$, we see that

$$
\frac{df_{n,p}}{dx_4}\left(\frac{2(p+1)}{n}\right) = -\frac{8(p+1)(n-p+1)\left(n^2-4np-4n+2p^2-2p-4\right)}{n^2}.
$$

For $\frac{n}{2} \leq p \leq n-1$, we see that

$$
\begin{aligned}
& n^2 - 4np - 4n + 2p^2 - 2p - 4 = -\frac{n^2}{2} + 2\left(p - \frac{n}{2}\right)^2 + (-2n-2)\left(p - \frac{n}{2}\right) - 5n - 4 \\
& \leq -\frac{n^2}{2} + 2\left(n - 1 - \frac{n}{2}\right)^2 + (-2n-2)\left(p - \frac{n}{2}\right) - 5n - 4 \\
& = (-2n-2)\left(p - \frac{n}{2}\right) - 7n - 2 < 0,
\end{aligned}
$$

and thus we have $\frac{df_{n,p}}{dx_4}\left(\frac{2(p+1)}{n}\right) > 0$. Hence, for $\frac{n}{2} \leq p \leq n-1$, the equation $\frac{df_{n,p}}{dx_4}(x_4) = 0$ has a real solution u_1 with

$$
\frac{2(p-1)}{2n} < u_1 < \frac{2(p+1)}{n}.
$$

We claim that the polynomial $\frac{df_{n,p}}{dx_4}$ of degree 3 is monotone increasing, and hence $f_{n,p}$ attains a local minimum at $x_4 = u_1$.

Now the second derivative of $f_{n,p}$ is given by

$$
\begin{aligned}
& \frac{d^2 f_{n,p}}{d{x_4}^2}(x_4) = 12n(n+1)(2n-p+1){x_4}^2 - 24(n+1)\left(n^2 + 2np + n - p^2 + p\right) x_4 \\
& 4\left(n^3 + n^2(9p+7) + 4n\left(p^2 + 4p + 2\right) - 2(p+1)\left(p^2 - 2p - 1\right)\right).
\end{aligned}
$$

To see that the second derivative of $f_{n,p}$ is positive for $\frac{n}{2} \leq p \leq n-1$, we note that the discriminant of the polynomial $\frac{d^2 f_{n,p}}{d{x_4}^2}$ of degree 2 is given by

$$
\begin{aligned}
& \left(12(n+1)\left(n^2 + 2np + n - p^2 + p\right)\right)^2 - 12n(n+1)(2n-p+1) \times \\
& 4\left(n^3 + n^2(9p+7) + 4n\left(p^2 + 4p + 2\right) - 2(p+1)\left(p^2 - 2p - 1\right)\right) \\
& = 48(n+1)\left(n^5 - 5n^4 p - 6n^4 + 7n^3 p^2 - 4n^3 p - 14n^3 - 4n^2 p^3 + 20n^2 p^2 + 4n^2 p\right. \\
& \left. -9n^2 + np^4 - 14np^3 + 13np^2 + 2np - 2n + 3p^4 - 6p^3 + 3p^2\right).
\end{aligned}
$$

We put

$$
\begin{aligned}
h_{n,p} = & \left(n^5 - 5n^4 p - 6n^4 + 7n^3 p^2 - 4n^3 p - 14n^3 - 4n^2 p^3 + 20n^2 p^2 + 4n^2 p\right. \\
& \left. -9n^2 + np^4 - 14np^3 + 13np^2 + 2np - 2n + 3p^4 - 6p^3 + 3p^2\right).
\end{aligned}
$$

We consider $h_{n,p}$ as a polynomial of p and we show that $h_{n,p} < 0$ for $\frac{n}{2} \leq p \leq n-1$. We have

$$
\begin{aligned}
h_{n,p}(p) = & (n+3)p^4 - 2(n+3)(2n+1)p^3 + \left(7n^3 + 20n^2 + 13n + 3\right) p^2 \\
& -n\left(5n^3 + 4n^2 - 4n - 2\right) p + n(n+1)\left(n^3 - 7n^2 - 7n - 2\right),
\end{aligned}
$$

$$
\begin{aligned}
\frac{dh_{n,p}}{dp}(p) = & 4(n+3)p^3 - 6(n+3)(2n+1)p^2 \\
& +2\left(7n^3 + 20n^2 + 13n + 3\right) p - n\left(5n^3 + 4n^2 - 4n - 2\right)
\end{aligned}
$$

and

$$\frac{d^2h_{n,p}}{dp^2}(p) = 12(n+3)p^2 - 12(n+3)(2n+1)p + 2\left(7n^3+20n^2+13n+3\right)$$
$$= 12(n+3)(-n+p+1)^2 - 36(n+3)(-n+p+1) + 2\left(n^3-4n^2+7n+39\right)$$
$$= 12(n+3)(-n+p+1)^2 - 36(n+3)(-n+p+1)$$
$$+2(n-2)^3+4(n-2)^2+6(n-2)+90>0$$

for $n\geq 2$ and $p\leq n-1$.

Thus $\dfrac{dh_{n,p}}{dp}$ is a monotone increasing function, and we see that

$$\frac{dh_{n,p}}{dp}\left(\frac{2}{3}n\right) = \frac{1}{27}n\left(5n^3+204n^2+360n+162\right)>0,$$

$$\frac{dh_{n,p}}{dp}\left(\frac{1}{2}n\right) = -\frac{1}{16}n\left(3n^4+73n^3+152n^2+116n+32\right)<0.$$

Thus the equation $\dfrac{dh_{n,p}}{dp}=0$ has a unique solution α with $\dfrac{1}{2}n<\alpha<\dfrac{2}{3}n$ and the function $h_{n,p}$ attains the minimum only at $p=\alpha$.

Note that $h_{n,p}\left(\dfrac{n}{2}\right) = -\dfrac{1}{16}n\left(3n^4+73n^3+152n^2+116n+32\right)<0$ and $h_{n,p}(n-1) = -2\left(n^4+4n^3+10n^2+6n-6\right)<0$. Thus we get that $h_{n,p}<0$ for $\dfrac{n}{2}\leq p\leq n-1$.

Since $f_{n,p}(0)=8(p+1)^2(n+p+1)>0$, in order to show that $f_{n,p}(x_4)>0$ for $x_4>0$, we need to prove that the local minimum $f_{n,p}(u_1)$ is positive.

We consider the tangent lines l_1, l_2, l_3 of the curve $f_{n,p}(x_4)$ at the points P_1 with x-coordinate $x_4=2(p-1)/n$, P_2 with $x_4=2(p+1)/n$, and P_3 with $x_4=2p/n$. The equation of the line l_1 is given by

$$l_1(t) = \frac{16}{n^3}(n-p+1)^2\left(2n^2+4n+p^3-p^2-p+1\right)$$
$$-\frac{8}{n^2}(n-p+1)\left(2n^3-3n^2p+13n^2-8np+16n+2p^3-6p+4\right)\left(t-\frac{2(p-1)}{n}\right),$$

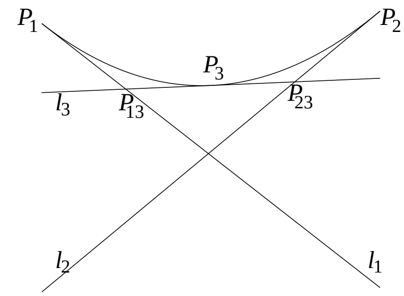
the equation of the line l_2 is given by

$$l_2(t) = \frac{16}{n^3}(p+1)^3(n-p+1)^2$$
$$-\frac{8}{n^2}(p+1)(n-p+1)\left(n^2-4np-4n+2p^2-2p-4\right)\left(t-\frac{2(p+1)}{n}\right),$$

and the equation of the line l_3 is given by

$$l_3(t) = \frac{8}{n^3}\left(n^4-n^3p+n^3+2n^2p^3-2n^2p-4np^4+2np^3+2np^2+2p^5-2p^4\right)$$
$$-\frac{8}{n^2}(n-p+1)\left(n^3-n^2p+n^2-2np^2-2np+2p^3\right)\left(t-\frac{2p}{n}\right).$$

Let P_{13} be the point at which the tangent lines l_1, l_3 intersect and P_{23} the point at which the tangent lines l_2, l_3 intersect. The coordinates (α_1, β_1) of the point P_{13} are given by Let P_{13} be the point at which the tangent lines l_1, l_3 intersect and P_{23} the point at which the tangent lines l_2, l_3 intersect. The coordinates (α_1, β_1) of the point P_{13} are given by



$$
\begin{aligned}
&\alpha_1 = \Big(\big(2p^2+3p-1\big)(n-p)^3 + \big(2p^3+15p^2+13p+4\big)(n-p)^2 \\
&+ \big(2p^4+15p^3+39p^2+36p+12\big)(n-p) + (2p+1)(n-p)^4 + 2p^4+13p^3 \\
&+28p^2+22p+6\big) / (n(n-p+1)\big((n-p)^3+p(n-p)^2+\big(p^2+6p+4\big)(n-p) \\
&+(p+2)\big(p^2+4p+2\big)\big)\big), \\
&\beta_1 = \big(8(p+1)\big(n^6+2n^5p^2-3n^5p+n^5-8n^4p^3+6n^4p^2-2n^4p-8n^4+14n^3p^4 \\
&+4n^3p^3+18n^3p^2+14n^3p-14n^3-12n^2p^5-16n^2p^4-4n^2p^3+28n^2p^2+26n^2p \\
&-6n^2+4np^6+8np^5-24np^4-28np^3+12np^2+12np+12p^5-12p^3\big) / \\
&\big(n^3\big((n-p)^3+p(n-p)^2+\big(p^2+6p+4\big)(n-p)+(p+2)\big(p^2+4p+2\big)\big)\big) \\
&= \big(8(p+1)\big((n-p)^6+\big(2p^2+3p+1\big)(n-p)^5+\big(2p^3+6p^2+3p-8\big)(n-p)^4 \\
&+\big(2p^4+18p^3+20p^2-18p-14\big)(n-p)^3+\big(2p^5+17p^4+48p^3+22p^2-16p-6\big)\times \\
&(n-p)^2+\big(3p^5+19p^4+38p^3+22p^2\big)(n-p)+p^2\big(p^3+6p^2+12p+6\big)\big)\big) / \\
&\big(n^3\big((n-p)^3+p(n-p)^2+\big(p^2+6p+4\big)(n-p)+(p+2)\big(p^2+4p+2\big)\big)\big)
\end{aligned}
$$

and the coordinates (α_2, β_2) of the point P_{23} are given by

$$
\begin{aligned}
&\alpha_2 = \big((2p-1)(n-p)^4+\big(2p^2+25p-15\big)(n-p)^3 \\
&+\big(2p^3+37p^2+35p-36\big)(n-p)^2+\big(2p^4+13p^3+49p^2+4p-28\big)(n-p) \\
&+2p^4+11p^3+12p^2-6p-6\big) / \big(n(n-p+1)\big((n-p)^3+(p+12)(n-p)^2 \\
&+\big(p^2+18p+16\big)(n-p)+(p+2)\big(p^2+4p+2\big)\big)\big), \\
&\beta_2 = 8\big(2n^7-7n^6p+29n^6+2n^5p^3+9n^5p^2-66n^5p+79n^5-8n^4p^4+26n^4p^3 \\
&+28n^4p^2-178n^4p+84n^4+14n^3p^5-78n^3p^4+106n^3p^3+72n^3p^2-192n^3p+38n^3 \\
&-12n^2p^6+68n^2p^5-172n^2p^4+144n^2p^3+82n^2p^2-84n^2p+6n^2+4np^7-20np^6 \\
&+88np^5-148np^4+64np^3+24np^2-12np-12p^6+36p^5-36p^4+12p^3\big) / \\
&\big(n^3\big((n-p)^3+(p+12)(n-p)^2+\big(p^2+18p+16\big)(n-p)+(p+2)\big(p^2+4p+2\big)\big)\big) \\
&= 8\big(2(n-p)^7+(7p+29)(n-p)^6+\big(2p^3+9p^2+108p+79\big)(n-p)^5 \\
&+\big(2p^4+36p^3+133p^2+217p+84\big)(n-p)^4 \\
&+\big(2p^5+46p^4+138p^3+150p^2+144p+38\big)(n-p)^3 \\
&+\big(2p^6+17p^5+89p^4+82p^3+10p^2+30p+6\big)(n-p)^2 \\
&+3\big((p-1)^4+10(p-1)^3+37(p-1)^2+44(p-1)+6\big)p^2(n-p) \\
&+\big((p-1)^4+9(p-1)^3+23(p-1)^2+13(p-1)-8\big)p^2\big) / \\
&\big(n^3\big((n-p)^3+(p+12)(n-p)^2+\big(p^2+18p+16\big)(n-p)+(p+2)\big(p^2+4p+2\big)\big)\big).
\end{aligned}
$$

Note that β_1, β_2 are positive for $1 \le p \le n-1$.

Since $f_{n,p}(x_4)$ is concave up, we see that the curve $(x_4, f_{n,p}(x_4))$ for $(\dfrac{2(p-1)}{n} \leq x_4 \leq \dfrac{2p}{n})$ lies inside the triangle given by the three points P_1, P_{13} and P_3, and that the curve $(x_4, f_{n,p}(x_4))$ for $(\dfrac{2p}{n} \leq x_4 \leq \dfrac{2(p+1)}{n})$ lies inside the triangle given by the three points P_3, P_{23} and P_2. Since the point $(u_1, f_{n,p}(u_1))$ is inside of one of these triangles, we see that the local minimum $f_{n,p}(u_1)$ is positive for $n/2 \leq p \leq n-1$, and thus we get our claim.

Case 2. We obtain the equation

$$x_1x_4(n-p+1)+x_3x_4(n-p+1)-2(n+1)x_2x_4+(p+1)x_1x_2+(p+1)x_2x_3=0,$$

we put $x_1=1$ and we get a following system of equations

$$\begin{aligned}
&-(n+p+1){x_2}^2x_4-(n-3p+1){x_3}^2x_4+(n+p+1)x_4+4(n+1)x_2x_3x_4\\
&-4(n+1)x_3x_4-(p+1)x_2{x_3}^2-(p+1)x_2{x_4}^2+(p+1)x_2=0, &(11)\\
&-(n-p+1){x_2}^2x_4+(2n-p+1)x_2{x_3}^2-(2n-p+1)x_2{x_4}^2+(2n-3p-1)x_2\\
&+(n-p+1){x_3}^2x_4-(n-p+1)x_4-4(n+1)x_2x_3+4(n+1)x_2x_4=0, &(12)\\
&x_4(n-p+1)+x_3x_4(n-p+1)-2(n+1)x_2x_4+(p+1)x_2+(p+1)x_2x_3=0 &(13)
\end{aligned}$$

From equation (13), we have

$$x_2=\frac{(x_3+1)x_4(n-p+1)}{2(n+1)x_4-(p+1)(x_3+1)}. \qquad (14)$$

Now we substitute equation (14) into equations (11) and (12), and we obtain the following equations:

$$\begin{aligned}
&F_1(x_3,x_4)=-\left(3n^3+5n^2p+9n^2+2np^2+12np+10n-2p^3+6p+4\right){x_3}^2{x_4}^2\\
&+2\left(5n^3+3n^2p+15n^2-2np^2+4np+14n+2p^3+2p+4\right)x_3{x_4}^2\\
&-\left(3n^3+5n^2p+9n^2+2np^2+12np+10n-2p^3+6p+4\right){x_4}^2\\
&+2(n+1)(p+1)(n+3p+1){x_3}^3x_4+4(p+1)^2(n-p+1){x_3}^3\\
&-2(n+1)(p+1)(5n-p+5){x_3}^2x_4+4(p+1)^2(2n-p+2){x_3}^2\\
&+2(n+1)(p+1)(n-p+1)x_3{x_4}^3-2(n+1)(p+1)(5n-p+5)x_3x_4\\
&+4(p+1)^2(n-p+1)x_3+2(n+1)(p+1)(n-p+1){x_4}^3\\
&+2(n+1)(p+1)(n+3p+1)x_4-2p(p+1)^2{x_3}^4-2p(p+1)^2=0, \qquad (15)
\end{aligned}$$

$$\begin{aligned}
&F_2(x_3,x_4)=2(p+1)(n-p){x_3}^3-2(n+1)(2n-3p-1){x_3}^2x_4\\
&-\left(3n^2+4np+8n-2p^2+2p+4\right){x_4}^2-\left(3n^2+4np+8n-2p^2+2p+4\right)x_3{x_4}^2\\
&-2(p+1)(n+p+2){x_3}^2+4(n+1)(2n+p+3)x_3x_4-2(p+1)(n+p+2)x_3\\
&+2(n+1)(2n-p+1){x_4}^3-2(n+1)(2n-3p-1)x_4+2(p+1)(n-p)=0. \quad (16)
\end{aligned}$$

We consider the resultant of the polynomials $F_1(x_3, x_4)$ and $F_2(x_3, x_4)$ with respect to x_4, which is a polynomial of x_3, say $P(x_3)$. We factor $P(x_3)$ as

$$\begin{aligned} P(x_3) &= -16(n+1)^4(p+1)^2(x_3+1)^4(n-p+1)^2(nx_3-n-2p-2)\times \\ &(nx_3-3n+2p-2)(x_3(3n-2p+2)-n)(x_3(n+2p+2)-n)\times \\ &\left(n^2(3n+4){x_3}^4-8n\left(2n^2+np+5n-p^2+3\right){x_3}^3\right. \\ &2\left(13n^3+8n^2p+36n^2-8np^2+16np+40n-16p^2+16\right){x_3}^2 \\ &\left.-8n\left(2n^2+np+5n-p^2+3\right)x_3+n^2(3n+4)\right). \end{aligned}$$

We denote by $Q_{n,p}(x_3)$ the factor of degree 4 in the above factorization:

$$\begin{aligned} Q_{n,p}(x_3) &= n^2(3n+4){x_3}^4-8n\left(2n^2+np+5n-p^2+3\right){x_3}^3 \\ &2\left(13n^3+8n^2p+36n^2-8np^2+16np+40n-16p^2+16\right){x_3}^2 \\ &-8n\left(2n^2+np+5n-p^2+3\right)x_3+n^2(3n+4). \end{aligned}$$

Now we consider two cases:

(a) the case when

$(nx_3-n-2p-2)(nx_3-3n+2p-2)(x_3(3n-2p+2)-n)(x_3(n+2p+2)-n)=0,$

(b) the case when $Q_{n,p}(x_3)=0$.

We claim that we get only Kähler-Einstein metrics on $Sp(n)/(U(p)\times U(n-p))$ in case (a).

1) The case when $x_3=\dfrac{n+2p+2}{n}$. In this case equations (15) and (16) are given by

$$\begin{aligned} &\frac{4}{n^4}(n+p+1)(nx_4-2p-2)(2(p+1)(n+p+1)-n(n-p+1)x_4)\times \\ &\left((-n-p-1)\left(n^2+2n-2p^2-2p\right)-n(n+1)(p+1)x_4\right)=0, \\ &\frac{2}{n^3}(nx_4-2p-2)\left(n^2(n+1)(2n-p+1){x_4}^2-n\left(3n^3+3n^2p+7n^2+4np^2\right.\right. \\ &\left.\left.+10np+6n-2p^3+2p^2+6p+2\right)x_4+2(p+1)(n+p+1)\left(n^2+2p^2+2p\right)\right)=0. \end{aligned}$$

If $nx_4-2p-2\neq 0$, we have

$$\begin{aligned} &(2(p+1)(n+p+1)-n(n-p+1)x_4)\times \\ &\left((-n-p-1)\left(n^2+2n-2p^2-2p\right)-n(n+1)(p+1)x_4\right)=0, \\ &\left(n^2(n+1)(2n-p+1){x_4}^2-n\left(3n^3+3n^2p+7n^2+4np^2+10np+6n-2p^3\right.\right. \\ &\left.\left.+2p^2+6p+2\right)x_4+2(p+1)(n+p+1)\left(n^2+2p^2+2p\right)\right)=0. \end{aligned}$$

By taking the resultant of these polynomials with respect to x_4, we get

$$-8n^6(n+1)^3(p+1)(n-2p)(n-p+1)(n+p+1)^4\left(n^2+2np+4n-2p^2+2\right),$$

and we see that the resultant is non-zero for $1\leq p\leq n-1$ and $n\neq 2p$. Thus we get only $x_4=\dfrac{2p+2}{n}$ for a solution of equations (15) and (16). From (14), we see $x_2=\dfrac{2(n+p+1)}{n}$. For $n=2p$, we get $x_4=\dfrac{p+1}{p}$ and $x_4=\dfrac{3p+1}{p}$ as solutions of equations (15) and (16). From (14), we see $x_2=\dfrac{3p+1}{p}$ and $x_2=\dfrac{p+1}{p}$ respectively. Thus we get Kähler-Einstein metrics in this case.

2) The case when $x_3 = \dfrac{3n-2p+2}{n}$. In this case equations (15) and (16) are given by

$$\begin{aligned}
&\frac{4}{n^4}(p+1)(n-p+1)(nx_4-4n+2p-2)\left(n^2(n+1)(2n-p+1){x_4}^2\right.\\
&+n\left(4n^3-11n^2p+n^2+10np^2-10np-4n-2p^3+6p^2-2p-2\right)x_4\\
&\left.-2(p+1)(2n-p+1)\left(3n^2-4np+2n+2p^2-2p\right)\right)=0,\\
&\frac{2}{n^3}(2n-p+1)(nx_4-4n+2p-2)\times\\
&(nx_4-2p-2)\left((n+1)nx_4+n^2-4np+2p^2-2p\right)=0.
\end{aligned}$$

If $nx_4-4n+2p-2 \neq 0$, we have

$$\begin{aligned}
&n^2(n+1)(2n-p+1){x_4}^2+n\left(4n^3-11n^2p+n^2+10np^2-10np-4n-2p^3\right.\\
&\left.+6p^2-2p-2\right)x_4-2(p+1)(2n-p+1)\left(3n^2-4np+2n+2p^2-2p\right)=0,\\
&(nx_4-2p-2)\left((n+1)nx_4+n^2-4np+2p^2-2p\right)=0.
\end{aligned}$$

By taking the resultant of these polynomials with respect to x_4, we get

$$8n^6(n+1)^3(p+1)(n-2p)(n-p+1)\left(n^2+2np+4n-2p^2+2\right),$$

and we see that the resultant is non-zero for $1 \leq p \leq n-1$ and $n \neq 2p$. Thus we get only $x_4 = \dfrac{4n-2p+2}{n}$ for a solution of equations (15) and (16). From (14), we see $x_2 = \dfrac{2(n-p+1)}{n}$. For $n=2p$, we get $x_4 = \dfrac{p+1}{p}$ and $x_4 = \dfrac{3p+1}{p}$ as solutions of equations (15) and (16). From (14), we see $x_2 = \dfrac{3p+1}{p}$ and $x_2 = \dfrac{p+1}{p}$ respectively. Thus we get Kähler-Einstein metrics in this case.

3) The case when $x_3 = \dfrac{n}{n+2p+2}$. In this case equations (15) and (16) are given by

$$\begin{aligned}
&\frac{4(n+p+1)}{(n+2p+2)^4}((n+2p+2)x_4-2(p+1))\times\\
&(2(p+1)(n+p+1)-(n-p+1)(n+2p+2)x_4)\times\\
&\left(-n^3-n^2p-3n^2+2np^2-2n+2p^3+4p^2+2p-(n+1)(p+1)(n+2p+2)x_4\right)=0,\\
&\frac{2}{(n+2p+2)^3}((n+2p+2)x_4-2(p+1))\cdot\left(2(p+1)(n+p+1)\left(n^2+2p^2+2p\right)\right.\\
&-(n+2p+2)\left(3n^3+3n^2p+7n^2+4np^2+10np+6n-2p^3+2p^2+6p+2\right)x_4\\
&\left.+(n+1)(2n-p+1)(n+2p+2)^2{x_4}^2\right)=0.
\end{aligned}$$

If $(n+2p+2)x_4-2(p+1) \neq 0$, we have

$$\begin{aligned}
&(2(p+1)(n+p+1)-(n-p+1)(n+2p+2)x_4)\times\\
&\left(-n^3-n^2p-3n^2+2np^2-2n+2p^3+4p^2+2p-(n+1)(p+1)(n+2p+2)x_4\right)=0,\\
&\left(2(p+1)(n+p+1)\left(n^2+2p^2+2p\right)-(n+2p+2)\left(3n^3+3n^2p+7n^2+4np^2\right.\right.\\
&\left.\left.+10np+6n-2p^3+2p^2+6p+2\right)x_4+(n+1)(2n-p+1)(n+2p+2)^2{x_4}^2\right)=0.
\end{aligned}$$

By taking the resultant of these polynomials with respect to x_4, we get

$$-8n^2(n+1)^3(p+1)(n-2p)(n-p+1)(n+p+1)^4(n+2p+2)^4\left(n^2+2np+4n-2p^2+2\right),$$

and we see that the resultant is non-zero for $1 \leq p \leq n-1$ and $n \neq 2p$. Thus we get only $x_4 = \dfrac{2(p+1)}{n+2p+2}$ for a solution of equations (15) and (16). From (14), we see $x_2 = \dfrac{2(n+p+1)}{n+2p+2}$. For $n = 2p$, we get $x_4 = \dfrac{p+1}{2p+1}$ and $x_4 = \dfrac{3p+1}{2p+1}$ as solutions of equations (15) and (16). From (14), we see $x_2 = \dfrac{3p+1}{2p+1}$ and $x_2 = \dfrac{p+1}{2p+1}$ respectively. Thus we get Kähler-Einstein metrics in this case.

4) The case when $x_3 = \dfrac{n}{3n-2p+2}$. In this case equations (15) and (16) are given by

$$
\begin{aligned}
&\frac{4(p+1)(n-p+1)}{(3n-2p+2)^4}((3n-2p+2)x_4 - 2(2n-p+1)) \cdot (-2(p+1)(2n-p+1)\times \\
&\left(3n^2-4np+2n+2p^2-2p\right) + (3n-2p+2)\left(4n^3-11n^2p+n^2+10np^2-10np\right. \\
&\left.-4n-2p^3+6p^2-2p-2\right)x_4 + (n+1)(3n-2p+2)^2(2n-p+1){x_4}^2\big) = 0, \\
&\frac{2(2n-p+1)}{(3n-2p+2)^3}((3n-2p+2)x_4 - 2(2n-p+1))((3n-2p+2)x_4 - 2(p+1)) \times \\
&\left((n+1)(3n-2p+2)x_4 + n^2-4np+2p^2-2p\right) = 0.
\end{aligned}
$$

If $(3n-2p+2)x_4 - 2(2n-p+1) \neq 0$, we have

$$
\begin{aligned}
&(2(p+1)(n+p+1) - (n-p+1)(n+2p+2)x_4) \times \\
&\left(-n^3-n^2p-3n^2+2np^2-2n+2p^3+4p^2+2p-(n+1)(p+1)(n+2p+2)x_4\right) = 0, \\
&\big(2(p+1)(n+p+1)\left(n^2+2p^2+2p\right) - (n+2p+2)\left(3n^3+3n^2p+7n^2+4np^2\right. \\
&\left.+10np+6n-2p^3+2p^2+6p+2\right)x_4 + (n+1)(2n-p+1)(n+2p+2)^2{x_4}^2\big) = 0.
\end{aligned}
$$

By taking the resultant of these polynomials with respect to x_4, we get

$$
8n^2(n+1)^3(p+1)(n-2p)(3n-2p+2)^4(n-p+1)\left(n^2+2np+4n-2p^2+2\right),
$$

and we see that the resultant is non-zero for $1 \leq p \leq n-1$ and $n \neq 2p$. Thus we get only $x_4 = \dfrac{2(2n-p+1)}{3n-2p+2}$ for a solution of equations (15) and (16). From (14), we see $x_2 = \dfrac{2(n-p+1)}{3n-2p+2}$. For $n = 2p$, we get $x_4 = \dfrac{p+1}{2p+1}$ and $x_4 = \dfrac{3p+1}{2p+1}$ as solutions of equations (15) and (16). From (14), we see $x_2 = \dfrac{3p+1}{2p+1}$ and $x_2 = \dfrac{p+1}{2p+1}$ respectively. Thus we get Kähler-Einstein metrics in this case.

Now we consider the case (b), that is, the case when $Q_{n,p}(x_3) = 0$. We compute a Gröbner basis of $\{F_1(x_3, x_4),\ F_2(x_3, x_4),\ Q_{n,p}(x_3)\}$ using the lex order with $x_4 > x_3$. We can find the following polynomials in the Gröbner basis:

$$
\begin{aligned}
&Q_{n,p}(x_3), \\
&n^2(3n+4){x_3}^3 - n\left(19n^2+8np+44n-8p^2+24\right){x_3}^2 \\
&+3\left(29n^3+32n^2p+92n^2-24np^2+40np+96n-32p^2+2\right)x_3 \\
&-n\left(13n^2-8np+20n-8p+8\right) - 8n(n+1)(2n-p+1)x_4, \qquad (17) \\
&n{x_3}^2 - 2(n+2p+2)x_3 + n(x_3+1)x_4. \qquad (18)
\end{aligned}
$$

From (18) we have

$$x_4 = \frac{-n{x_3}^2 + 2x_3(n+2p+2) - n}{n(x_3+1)}. \tag{19}$$

We substitute equation (19) into equation (14), and we obtain

$$x_2 = \frac{(x_3+1)(-n+p-1)\left(-n{x_3}^2 + 2x_3(n+2p+2) - n\right)}{n(2n+p+3){x_3}^2 - 2\left(2n^2+3np+5n+4p+4\right)x_3 + n(2n+p+3)}. \tag{20}$$

Let a_k $(k = 0, \cdots, 4)$ denote the coefficients of the polynomial $Q_{n,\,p}(x_3)$. Then we have that $a_k = a_{4-k}$. Thus we see that if the equation $Q_{n,\,p}(x_3) = 0$ has a solution $x_3 = \alpha$, then so is $x_3 = \dfrac{1}{\alpha}$.

We claim that the equation $Q_{n,p}(x_3) = 0$ has four different positive solutions. It is enough to see that $Q_{n,p}(x_3) = 0$ has two different solutions between $0 < x_3 < 1$.

Note that $Q_{n,p}(0) = n^2(3n+4) > 0$ and $Q_{n,p}(1) = 32(p+1)(n-p+1) > 0$.

$$\begin{aligned}
&Q_{n,p}\left(\frac{1}{2}\right) = \frac{1}{16}\left(-5n^3 - 16n^2p - 44n^2 + 16np^2 + 128np + 80n - 128p^2 + 128\right)\\
&= -5(n-p-1)^3 + (-31p-59)(n-p-1)^2 + \left(-31p^2-22p-23\right)(n-p-1)\\
&-5p^3 - 75p^2 + 89p + 159\\
&= -5(n-p-1)^3 + (-31p-59)(n-p-1)^2 + \left(-31p^2-22p-23\right)(n-p-1)\\
&-5(p-2)^3 - 105(p-2)^2 - 271(p-2) - 3 < 0 \quad \text{for} \quad 2 \le p \le n-1.
\end{aligned}$$

For $p = 1$, we have

$$Q_{n,1}\left(\frac{1}{2}\right) = -\frac{1}{16}n\left(5n^2 + 60n - 224\right) = -\frac{1}{16}\left(5(n-3)^2 + 90(n-3) + 1\right)n < 0$$

for $n \ge 3$. Thus we get our claim.

Now, for a solution of the equation $Q_{n,p}(x_3) = 0$, we get values of x_4 and x_2 from (19) and (20), and thus we get four triples of solutions $\{x_2^0, x_3^0, x_4^0\}$ of the system of equations (11), (12), (13) in case (b).

We claim that only two triples of the solutions of the system of equations (11), (12), (13) have the property that $x_2^0 > 0$, $x_3^0 > 0$, $x_4^0 > 0$ in case (b).

Consider the resultant of the polynomials

$$Q_{n,p}(x_3) \;\text{ and }\; n{x_3}^2 - 2(n+2p+2)x_3 + n(x_3+1)x_4$$

with respect to x_3, which is a polynomial of x_4, say $S_{n,\,p}(x_4)$. Then we get that, up to a scalar multiple $32n^4(n+p+1)$,

$$\begin{aligned}
&S_{n,\,p}(x_4) = n^2(n+1)(3n+4)(2n-p+1){x_4}^4\\
&+4n(n+1)(2n-p+1)\left(n^2 - 4np - 2p^2 - 8p - 2\right){x_4}^3\\
&+2\left(n^5 - 19n^4p - 11n^4 + 36n^3p^2 - 18n^3p - 30n^3 + 22n^2p^3 + 130n^2p^2 + 54n^2p\right.\\
&\left.-22n^2 - 16np^4 + 4np^3 + 108np^2 + 68np - 4n - 16p^4 - 16p^3 + 16p^2 + 16p\right){x_4}^2\\
&-8(p+1)(n-2p)(n+p+1)\left(n^2 - 6np - 2n + 2p^2 - 4p - 2\right)x_4\\
&+8(p+1)^2(n-2p)^2(n+p+1).
\end{aligned}$$

From (20) we get a polynomial

$$\begin{aligned}
&P_{n,p}(x_2, x_3) = x_2 \cdot \left(n(2n+p+3){x_3}^2 - 2\left(2n^2+3np+5n+4p+4\right)x_3 + n(2n+p+3)\right)\\
&-(x_3+1)(-n+p-1)\left(-n{x_3}^2 + 2x_3(n+2p+2) - n\right).
\end{aligned}$$

We also consider the resultant of the polynomials $Q_{n,p}(x_3)$ and $P_{n,p}$ with respect to x_3, which is a polynomial of x_2, say $T_{n,p}(x_2)$. Then we get that, up to a scalar multiple $1024n^4(n+1)(n-p+1)^2(p+1)^2(n+p+1)^3$,

$$
\begin{aligned}
&T_{n,p}(x_2) = n^2(n+1)(3n+4)(n+p+1){x_2}^4\\
&-4n(n+1)(n+p+1)\left(5n^2-8np+8n+2p^2-8p+2\right){x_2}^3\\
&+2\left(24n^5-55n^4p+89n^4+6n^3p^2-190n^3p+116n^3+42n^2p^3+46n^2p^2-222n^2p\right.\\
&\left.+62n^2-16np^4+60np^3+60np^2-100np+12n-16p^4+16p^3+16p^2-16p\right){x_2}^2\\
&-8(n-2p)(n-p+1)(2n-p+1)\left(3n^2-2np+6n-2p^2-4p+2\right)x_2\\
&+8(n-2p)^2(n-p+1)^2(2n-p+1).
\end{aligned}
$$

Note that $T_{n,p}(x_2) = S_{n,n-p}(x_2)$ by a direct computation.

Now we claim that

(I) for $1 \leq p < \dfrac{n}{2}$, the equation $S_{n,p}(x_4) = 0$ has two different positive solutions and two different negative solutions

and

(II) for $\dfrac{n}{2} < p \leq n-1$, the equation $S_{n,p}(x_4) = 0$ has four different positive solutions.

Note that, for $x_4 = 0$, we have $S_{n,p}(0) = 8(p+1)^2(n-2p)^2(n+p+1) > 0$ for $p \neq \dfrac{n}{2}$, and for $x_4 = \dfrac{2(p+1)}{n}$, we have

$$S_{n,p}\left(\frac{2(p+1)}{n}\right) = -\frac{1}{n^2}16(p+1)^2(n-p+1)^2\left(n(p-1)+4p^2+4p\right) < 0.$$

For $x_4 = \dfrac{2(2p-n)}{n}$, we have

$$S_{n,p}\left(\frac{2(2p-n)}{n}\right) = \frac{8}{n}(n-2p)^2(n-p+1)^2\left(5n^2-9np+3n+4p^2-4p\right).$$

Note that

$$
\begin{aligned}
5n^2-9np+3n+4p^2-4p &= 5(n-p)^2+(p+3)(n-p)-p\\
&> (n-p)^2+(p+3)\cdot 1-p = (n-p)^2+3 > 0.
\end{aligned}
$$

Thus we get $S_{n,p}\left(\dfrac{2(2p-n)}{n}\right) > 0$.

Now for $x_4 = \dfrac{(2p-n)}{n}$, we have

$$
\begin{aligned}
&S_{n,p}\left(\frac{2p-n}{n}\right) = -\frac{(n-2p)^2}{n^2}\left(5n^4p+9n^4-4n^3p^2+3n^3p+33n^3+4n^2p^2\right.\\
&\left.-20n^2p+40n^2-20np^3+4np^2-32np+16n+16p^4-16p\right)\\
&= -\frac{(n-2p)^2}{n^2}\left(\left(16p^2+39p+33\right)(n-p)^3+\left(18p^3+67p^2+79p+40\right)(n-p)^2+\left(8p^4\right.\right.\\
&\left.\left.+33p^3+63p^2+48p+16\right)(n-p)+(5p+9)(n-p)^4+p^5+12p^4+17p^3+8p^2\right) < 0.
\end{aligned}
$$

For $1 \leq p < \dfrac{n}{2}$, we have $2p-n < 0$ and hence $2(2p-n) < 2p-n < 0 < 2(p+1)$. Thus we see that the equation $S_{n,p}(x_4) = 0$ has the four solutions x_4^1, x_4^2, x_4^3, x_4^4 with

$$\frac{2(2p-n)}{n} < x_4^1 < \frac{2p-n}{n} < x_4^2 < 0 < x_4^3 < \frac{2(p+1)}{n} < x_4^4.$$

For $\dfrac{n}{2} < p \leq n-1$, we have $2p-n > 0$. Since $2(2p-n)-2(p+1) = 2(p-n) < 0$, we have $0 < 2p-n < 2(2p-n) < 2(p+1)$. Thus we see that the equation $S_{n,p}(x_4) = 0$ has the four solutions x_4^1, x_4^2, x_4^3, x_4^4 with

$$0 < x_4^1 < \frac{2p-n}{n} < x_4^2 < \frac{2(2p-n)}{n} < x_4^3 < \frac{2(p+1)}{n} < x_4^4.$$

Noting that $T_{n,p}(x_2) = S_{n,n-p}(x_2)$, we also get that

(III) for $1 \leq p < \dfrac{n}{2}$, the equation $T_{n,p}(x_2) = 0$ has four different positive solutions.

and

(IV) for $\dfrac{n}{2} < p \leq n-1$, the equation $T_{n,p}(x_2) = 0$ has two different positive solutions and 2 different negative solutions.

Combining the statements (I), (II), (III), and (IV) we get exactly two non-Kähler Einstein metrics on $Sp(n)/(U(p)\times U(n-p))$ and this completes the proof.

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